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Spheres are rare

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Abstract

We prove that triangulations of homology spheres in any dimension grow much slower than general triangulations. Our bound states in particular that the number of triangulations of homology spheres in 3 dimensions grows at most like the power 1/3 of the number of general triangulations.

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1 Introduction

The "Gromov question" [1] asks whether in dimensions higher than 2 the number of triangulations of the sphere grows exponentially in the number of glued simplices, as happens in dimension 2, for which explicit formulas are known [2, 3, 4]. It has not been answered until now [5, 6, 7]. It is usually formulated for triangulations that are *homeomorphic* to a sphere. But we do not know counterexamples showing that such an exponential bound could not hold also more generally for *homology* spheres, although we are conscious that homotopy constraints are much stronger than homology constraints.

Understanding general triangulations is important in the quantum gravity [8, 9, 10] context. Recently a theory of general (unsymmetrized) random tensors of rank d was developped [11, 12, 13, 14, 15], with a new kind of $1/N$ expansion discovered [16, 17, 18]. This expansion is indexed by an integer, called the *degree*. It is not a topological invariant but a sum of genera of *jackets*, which are ribbon graphs embedded in the tensor graphs. It also allowed to find an associated critical behavior [19] and to discover and study new classes of renormalizable quantum field theories of the tensorial type [20, 21, 22, 23, 24].

In this note we perform a small step towards applying this new circle of ideas to the Gromov question. We prove a rather obvious result that we nevertheless could not find in the existing literature, namely that spherical triangulations are rare among all triangulations in any dimension. More precisely, we give in section 2 a necessary condition for a colored triangulation Γ to have a trivial homology. It states that the rank of the incidence matrix of edges and faces for the dual graph G of the triangulation Γ must be equal to the nullity of that graph (the number of edges not in a spanning tree). From this condition we deduce that any such graph has always at least one jacket of relatively low genus.

In section 3 we prove that ribbon graphs with such relatively low genus are quite rare among general graphs. Combining the two results proves the statement of the title.

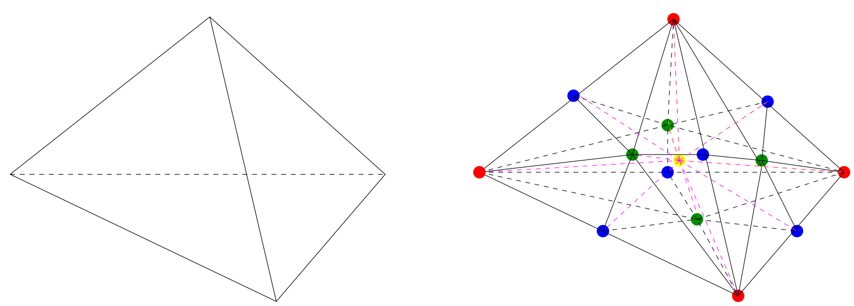
In particular in $d = 3$ our bound states that triangulations of homology spheres made of n tetrahedra grow at most as $(n!)^{1/3}$, while general triangulations made of n tetrahedra grow as $n!$ (up to K^n factors). Hence in dimension 3 spherical triangulations cannot grow faster than the cubic root of general triangulations.

An outlook of the connection with the tensor program for quantum grav-

ity is provided in the last section.

2 Spherical Triangulations

To any ordinary triangulation is associated a unique colored triangulation, namely its barycentric subdivision. If the initial triangulation is made of n simplices of dimension d (i.e. with $d+1$ summits), the barycentric subdivision is made of $a.n$ colored simplices, where a only depends on the dimension d. For instance in $d = 3$, a tetrahedron is decomposed by barycentric subdivision (see Figure below) into 24 colored tetrahedra, hence $a = 24$. This map is injective since the initial triangulation is made of the vertices of a single color, together with lines that correspond to given bicolored lines of the colored triangulation.



Hence in order to study the Gromov question, one can restricts oneself to *colored* triangulations. A bound in K^n for spherical colored triangulations in $d = 3$ would translate into a bound in K^{24n} for spherical ordinary triangulations and so on.

Colored triangulations [25, 26] triangulate pseudo-manifolds [12], and in contrast with the usual definition of spin foams in loop quantum gravity, they have a well-defined d-dimensional homology. They are in one-to-one correspondence with dual *edge-colored graphs*. Bipartite graphs correspond to triangulations of orientable pseudo-manifolds. Since they are associated to simple field theories [11] we expect they are the most natural objects for quantum gravity.

Therefore we consider from now on the category of (vacuum) connected bipartite edge-colored graphs with $d + 1$ colors and uniform coordination $d + 1$ (one edge of every color) at each vertex. The Gromov question can be rephrased as whether the number of such graphs with n vertices dual to

spherical colored triangulations is bounded by K^n.

We call V, E and F the set of vertices, edges and faces of the graph G. Faces are simply defined as the two-colored connected components of the graph, hence come in $d(d+1)/2$ different types, the number of pairs of colors. We also put $|V| = n$. n is the order of the graph and is even since the graph is bipartite. Also the graph being bipartite is naturally directed (i.e. there is a canonical orientation of each edge). We write $\mathcal{T}$ for a generic spanning tree of G and $G/\mathcal{T}$ for the contracted graph with one vertex also called the *rosette* associated to G and $\mathcal{T}$. The nullity of G (number of loops, namely number of edges in any rosette) is $|L| = |E| - |V| + 1 = 1 + n(d-1)/2$. Jackets are ribbon graphs passing through all vertices of the graph. There are $(d!)/2$ such jackets in dimension d [17]. Any jacket of genus g provides a Hegaard decomposition of the triangulated pseudo-manifold. In dimension 3 it gives a decomposition into two handle-bodies bounded by a common genus g surface [27].

The edge-face incidence matrix ε_{ef} describes the incidence relation between edges and faces. Let us orient each face arbitrarily: ε_{ef} is then +1, -1 or 0 depending on wether the face goes through the edge in the direct sense, opposite sense or does not go through e.[1]

Group field theory [28, 29, 30] can be used to write connections on G with a structure group $\mathcal{G}$. To each edge of a group field theory graph G is associated a generator $h_e \in \mathcal{G}$ representing parallel transport along e. The curvatures of the connection is the family of group elements $\left[\overrightarrow{\prod_{e\in f} h_e{}^{\varepsilon_{ef}}}\right]$, for all faces $f \in F$, where the product is taken in the right order of the face.

The generators h_e for the edges e in any given spanning tree $\mathcal{T}$ of G are irrelevant for the computation of $\pi_1(G)$, as they can be fixed to 1 through the usual $G^{|V|}$ gauge invariance on connections. Indeed setting $h_e = 1\ \forall e \in \mathcal{T}$ is equivalent to consider the retract $G/\mathcal{T}$ of G with a single vertex, hence the rosette associated to G and $\mathcal{T}$.[2]

The fundamental group $\pi_1(G)$ of G admits then a presentation with one such generator h_e per edge of $G/\mathcal{T}$ and the relations

$$\left[\overrightarrow{\prod_{e\in f}} h_e{}^{\varepsilon_{ef}}\right] = 1 \quad \forall f \in F. \tag{2.1}$$

[1]Faces running several times through an edge are excluded in colored graphs.

[2]After this fixing the gauge transformations are reduced to a single global conjugation of all remaining h_e by G.

Hence the space of flat connections (for which curvature is 1 for all faces) is the representation variety of $\pi_1(G)$ into G [31].

In particular G is simply connected if and only if the set of flat connections is just a point hence if the set of equations (2.1) has $h_e = 1\ \forall e \in G/\mathcal{T}$ as its unique solution.

The homology of G is even simpler, as it corresponds to the case of a commutative group $\mathcal{G}$. G has trivial first-homology (i.e $H_1(G) = 0$) if and only if the set of commutative equations

$$\sum_{e \in L = G/\mathcal{T}} \varepsilon_{ef} h_e = 0 \quad \forall f \in F \tag{2.2}$$

have $h_e = 0$ as unique solution.

By (commutative) gauge invariance the rank r_G of the matrix ε_{ef} is equal to the rank of the reduced matrix $\varepsilon^{\mathcal{T}}_{ef}$ where the edges e run over the reduced set of edges of $G/\mathcal{T}$. We have certainly

$$r_G \leq \inf\{|L|, |F|\}. \tag{2.3}$$

Lemma 1. *The edge-colored graph G has trivial first-homology if and only if the rank of $\varepsilon^{\mathcal{T}}_{ef}$ is maximal, i.e. equal to $|L| = 1 + (d-1)n/2$. This writes:*

$$r_G = 1 + (d-1)n/2 \tag{2.4}$$

Proof. The rank r_G cannot be larger than $|L|$ by (2.3). If it is strictly smaller it would mean that the linear map from $\mathbb{R}^L$ to $\mathbb{R}^F$ represented by the matrix ε_{ef} would have a non trivial kernel, hence the relations (2.2) defining the (first) homology space $H_1(G)$ would have non-trivial solutions. $\square$

Remark that the rank condition (2.4) implies that $|F|$ must be at least $|L|$ hence at least $1 + (d-1)n/2$.

In dimension d we have $(d!)/2$ jackets J and for each of them the relation

$$2 - 2g(J) = n - |E| + F_J \implies g(J) = 1 + (d-1)n/4 - F_J/2 \tag{2.5}$$

Since each face belongs to exactly $(d-1)!$ jackets we have $\sum_J F_J = (d-1)!|F|$ and the degree of the graph is

$$\omega(G) = \sum_J g(J) = (d-1)![d/2 + d(d-1)n/8 - |F|/2] \tag{2.6}$$

It means that for a graph G with $H_1(G) = 0$, the degree obeys the bound

$$0 \leq \omega(G) \leq (d-1!)[(d-1)/2 + (d-1)(d-2)n/8]. \tag{2.7}$$

In dimension d, duality exchanges the k-th and $(d-k)$-th Betti numbers.

Lemma 2. *In a graph G with $H_1(G) = 0$, hence dual to a triangulation Γ such that $H_{d-1}(\Gamma) = 0$, there exists at least one jacket J_0 whose genus is bounded by*

$$g(J_0) \leq \frac{d-1}{d}[1 + \frac{(d-2)n}{4}] \tag{2.8}$$

Proof. We just divide the bound (2.7) by the number $(d!)/2$ of the jackets. $\square$

Since spheres have trivial homologies hence zero Betti numbers between 1 and $d-1$, it follows that graphs dual to spherical triangulations all obey Lemma 2. Of course triangulations of true (homotopy) spheres could be much rarer but we won't investigate this question here.

3 Low Genus Bounds

In the previous section we proved that colored graphs dual to spherical triangulations must have at least one jacket of relatively low genus. Let us now exploit this condition to bound the number of such graphs. They are Feynman graphs and occur with their correct weights in the perturbative expansion of a random tensor theory with action

$$e^{\lambda T_0 \cdots T_d + \bar{\lambda}\bar{T}_0 \cdots \bar{T}_d - \sum_{i=0}^{d} T_i \bar{T}_i}. \tag{3.9}$$

where T and $\bar{T}$ are tensors whose indices are contracted according to the pattern of the complete graph on $d+1$ vertices. This is detailed at length in [11, 13].

Consider now a particular jacket J_0. Suppressing all strands not in that jacket reduces the tensorial action to a matrix action of the type

$$e^{\lambda \mathrm{Tr} M_0 \cdots M_d + \bar{\lambda} \mathrm{Tr} \bar{M}_0 \cdots \bar{M}_d - \sum_{i=0}^{d} \mathrm{Tr} M_i M_i^\dagger}. \tag{3.10}$$

We know from Euler's formula that the corresponding ribbon Feynman graphs with n vertices have genus g bounded by $g \leq g_{max}(n) = I(\frac{2+n(d-1)}{4})$, where I means "integer part". For $d = 3$ this means that the genus of a

bipartite ribbon graph of the ϕ^4 type is bounded by $n/2$, where n is the (even) order of the graph.

Let $T_{d,g,n}$ be the number of ribbon graphs of order n and genus g corresponding to the action (3.10). Our main bound is

Lemma 3. *There exists a constant K_d such that*

$$|T_{d,g,n}| \leq K_d^n n^{2g} \tag{3.11}$$

Proof. Because of the bipartite character of action (3.10), $n = 2p$ is even. We want to count the number of Wick contractions matching $4p$ fields and $4p$ anti-fields on p vertices and p anti-vertices giving rise to a ribbon graph of genus g. The edges of such a graph can always be decomposed into a spanning tree $\mathcal{T}$ of $n-1$ edges, a dual tree $\tilde{\mathcal{T}}$ in the dual graph made of $|F|-1 = n+1-2g$ edges, and a set of $2g$ "crossing edges" CE.[3] Paying an overall factor 3^{4n} we can preselect as $(A, \bar{A})$, $(B, \bar{B})$ and $(C, \bar{C})$ the fields and anti-fields which Wick-contract respectively into $\mathcal{T}$, $\tilde{\mathcal{T}}$ and CE. Building the Wick contractions between the $(n-1)$ fields of A and the $n-1$ anti-fields of $\bar{A}$ to form the labeled tree $\mathcal{T}$ certainly costs at most $(n-1)!$, the total number of such contractions. Contracting the tree $\mathcal{T}$ to a single vertex we obtain a cyclic ordering of the remaining $2(n+1)$ fields and anti-fields of $B \cup \bar{B} \cup C \cup \bar{C}$. Let us delete for the moment on the cycle the $2g$ fields of C and the $2g$ anti-fields of $\bar{C}$. Building the dual tree $\tilde{\mathcal{T}}$ out of contractions of the $n+1-2g$ fields of B and $n+1-2g$ antifields of $\bar{B}$ must create a new face per edge, hence the number of corresponding Wick contractions is bounded by the number of non-crossing matchings between B and $\bar{B}$ on the cycle. We know that the total number of such non-crossing matchings between $2p$ objects is the Catalan number $C_p \leq 4^p$, hence we obtain a bound 4^{n+1} for the Wick contractions of the fields of B with the anti-fields of $\bar{B}$. Finally the number of contractions joining the $2g$ fields of C to the $2g$ anti-fields of $\bar{C}$ to create the edges of CE can be bounded by joining them in any possible way, hence by $(2g)!$. Using the standard vertex symmetry factor $[p!]^{-2}$ of Feynman graphs coming from expanding the exponential action in (3.10) (and since $g \leq g_{max}(n) = I(\frac{2+n(d-1)}{4})$), we easily conclude that building $\mathcal{T}$, $\tilde{\mathcal{T}}$ and CE costs at most $K^n n^{2g}$ Wick contractions, and we get (3.11) with $K_d = 3^4 K$. □

[3]There are usually many different such decompositions but we just choose arbitrarily one of them for each graph.

Notice that we did not try at all to optimize K_d (in particular in the proof of the Lemma above we did not try to use the colors which give further constraints, as they would not improve on the factor n^{2g}. Remark also that the upper bound (3.11) of Lemma 3 does not contradict the well-established apparently larger asymptotic behavior [32]

$$T_{g,n} \simeq_{n\to\infty} c_g \cdot n^{5/2(g-1)} 12^n \tag{3.12}$$

for *fixed* g and large n which is e.g. used to define double scaling in matrix models. Indeed this asymptotic behavior cannot be maintained when g grows with n, as we know that $g \le n/2$.

Let us neglect fixed powers of n since they can be absorbed into new K^n factors. In dimension d general connected graphs at order n grow as $K^n n^{n(d-1)/2}$, as expected for a ϕ^{d+1} interaction. The number of graphs satisfying Lemma 2 on the other hand is bounded by $(d!)/2$ (to choose the jacket J_0) times the number of ribbon graphs with genus $g \le \frac{d-1}{d}[1 + \frac{(d-2)n}{4}]$ in that J_0 jacket. By Lemma 3 it is therefore bounded by $(K')^n \cdot n^{2g(J_0)}$, hence by $(K")^n \cdot n^{\frac{(d-1)(d-2)}{2d} n}$.

Putting together these results we obtain

Theorem 1. *There exist constants K and K' such that the number ST_n of spherical triangulations with n simplices is bounded by*

$$ST_N \le K^n n^{\frac{(d-1)(d-2)}{2d} n} \tag{3.13}$$

and such that the number of general triangulations T_n obeys

$$T_N \ge (K')^n n^{\frac{d-1}{2}} \tag{3.14}$$

Since in any dimension $\frac{(d-1)(d-2)}{2d} < \frac{d-1}{2}$, we have

$$\lim_{n\to\infty} ST_n/T_n = 0 \tag{3.15}$$

which means that triangulations of spheres are always rare among general triangulations.

In dimension 3 we get that triangulations of homology spheres grow at most as $(n!)^{1/3}$ whether general triangulations grow at least as $n!$. Hence spherical triangulations cannot grow faster than a cubic root of general triangulations.

4 Outlook

We would like a microscopic theory of quantum gravity to sum over all spaces irrespectively of their topology and to generate a macroscopic space-time such as the one we observe (large and of trivial topology). Since spheres are rare, this cannot be done without some non-trivial ponderation factor to favor them. Random tensor models have precisely such a factor; they ponder triangulations not by 1 but by $\lambda^{|V|}N^{|F|}$ where λ plays the role of the cosmological constant, N is the size of the tensor and $N^{|F|}$ is a discretization of the Einstein-Hilbert action. Indeed for flat equilateral triangulations, curvature is concentrated on the $d-2$ dimensional simplices, hence is associated to the faces of the dual graphs.

The $1/N$ expansion of random tensor models has melons (i.e. very particular "stacked" triangulations of the spheres) as their leading graphs. It can therefore lead from a perturbative phase around "no space at all" to the condensation of a primitive kind of space-time, namely the continuous random tree (CRT) [33], called branched polymer by physicists. Indeed melons are branched polymers [34]. This melonic CRT phase (of Hausdorff dimension 2) however *cannot be the end of the tensor story.* Indeed sub-melonic triangulations include e.g. at least all graphs planar in a fixed jacket. Such graphs are exactly well-labeled trees [4]. The labels create shortcuts on the CRT, leading to the very different 2D Brownian sphere phase (of Hausdorff dimension 4 [35]).[4] We hope that investigation of all submelonic contributions in higher rank random tensors will uncover similar but more complicated shortcuts on the melonic CRT. Ideally it could then lead through a sequence of phase transitions to an effective space-time similar to the one we observe, in which e.g. topological, spectral and Hausdorff dimensions all appear equal to 4.

However even if this is the case, many mysteries would remain. Let us briefly discuss one of them. The two dimensional phase transition from planar graphs to Brownian spheres or the higher-dimensional phase transition from melons to the continuous random tree occurs for the *unstable* sign of the coupling constant λ, where all graphs add up with the same sign. But we want to (Borel)-sum *all* triangulations (not only planar or melonic ones). Random tensors models can do that using the loop vertex expansion [36, 14, 37], but only for the *other sign* of the coupling constant, in which amplitudes

[4]They also realize a nice concrete toy model for the holographic principle, since all information about the faces of the triangulation is captured by the labels on the tree which has a single face as its boundary.

alternate with their order. It is also for this stable sign of the coupling constant that renormalizable tensor group field theories [20, 21, 22, 23, 24] have been proved asymptotically free [38], meaning the coupling constant λ in such theories grows naturally and should unavoidably reach some critical value and generate a phase transition.

It is tempting but difficult to integrate all these insights into a single coherent picture, ideally that of a renormalizable tensor group field theory whose renormalization group trajectory would lead through a sequence of phase transitions from no space at all to the 4D space we know of, equipped with general relativity as an effective theory. Indeed a major difficulty - underlined e.g. by Ambjorn [39]- is this incoherence of sign: Borel summability and asymptotic freedom require one sign of the coupling, when the CRT and Brownian sphere phase transitions occur for the other sign.[5] To connect both phenomena seems to require some kind of analytic continuation; let us simply remark that coupling constant flows which grow in the infrared are unstable under addition of an infinitesimal imaginary part, which can send them to the other side of the real axis. One could also speculate that it is perhaps at this point that it may be necessary to supplement the purely Euclidean approach with some form of continuation to a Lorentzian metric.

Answering the Gromov question is not a prerequisite for the tensor track program [40, 41], which proposes to ultimately (Borel)-sum over all triangulations anyway. However it is unclear whether submelonic corrections and the precise geometrogenetic phase transitions they could generate can be investigated in detail if we remain unable to answer the relatively simple and natural Gromov question. A positive answer would reinforce the analogy between planar graphs and random matrices on one side and spherical triangulations and random tensors on the other. It would lend some weight to the hope that e.g. the equally weighted measure on spherical triangulations in dimensions 3 and 4 could converge (in the Gromov-Hausdorff sense) to a new kind of compact random space generalizing the Brownian two-dimensional sphere [42, 43, 44] to higher dimensions. Also it would open the possibility that double or multiple scalings beyond the melonic graphs could be found *within* the spherical triangulations; in that case we would expect the result to be stable, in contrast to what happens in matrix models.

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[5] This difficulty also is the main reason for which double scaling in matrix models is unstable.

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